

NOTATION

$\theta_1, \theta_2, X, Y, Z, Fo$, dimensionless variables: temperatures, coordinates, and time; a, b , sides of rectangle; $a_1, a_2, \lambda_1, \lambda_2$, thermal diffusivities and thermal conductivities of wall material and fluid; γ_1, γ_2 , dimensionless semiaxes of ellipse; β , thickness of duct wall in Y direction; Pe, Bi , Péclet and Biot numbers; $W_z(X, Y)$ dimensionless velocity profile of fluid flow in duct.

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FUNDAMENTAL ASPECTS OF THE DEVELOPMENT OF ALGORITHMS FOR MATHEMATICAL MODELING OF THE THERMAL MODE OF THIN-WALLED STRUCTURES

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Specialized algorithms are proposed for computation of the temperature fields in thin-walled structural elements.

Among the universal methods of mathematical modeling of the thermal mode of a structure should be those based on solving systems of heat-conduction equations [1, 2]. Application of the method of "skeleton" structures [2] permits computing the temperature fields in structures of practically any geometry. Its universality lies in the fact that the "skeleton" structure combines the thermal models of the individual elements into a single generalized mathematical model. Moreover, it can also be used to compute the temperature fields in elements of complex geometry. For this, the element is partitioned into separate subdomains of canonical shape whose thermal state is described by the traditional heat-conduction equations. However, such a breaking down of the structural elements results in excessive awkwardness of the mathematical model and degrades its graphic appearance and convenience of application. Hence, the construction of typical methods and recipes for the solution of problems of analyzing temperature fields in groups of structural elements or individual elements of complex shape possessing definite characteristic criteria which would permit expansion of the domain of application of the method of "skeleton" structures is urgent. This paper is devoted to the development of algorithms to solve this problem.

The paper [3], in which an algorithm is proposed for the computation of temperature fields in thin-walled structural elements having the longitudinal coordinate z common for all plates, might be an example of the development of specialized algorithms.

Let us first examine the problem of computing the thermal state of the plates displayed in Fig. 1a. The temperature distribution in these plates is described by using the following system of heat-conduction equations

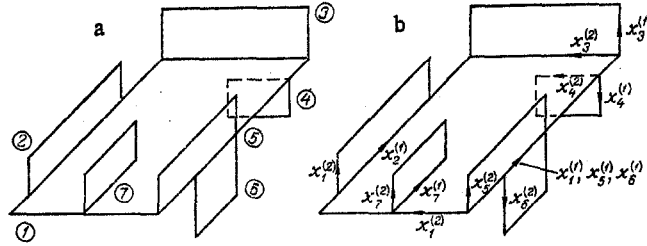


Fig. 1. Diagram of the plate connections (a) and the corresponding coordinate system (b).

$$\rho(x_j, T) C_p(x_j, T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_j} \left(\lambda(x_j, T) \frac{\partial T}{\partial x_j} \right) + q_v(x_j, T) + \frac{1}{\delta(x_j)} \lambda(x_i, T) \frac{\partial T}{\partial x_i} \Big|_{x_i \in \Gamma_i \cap D_j},$$

$$\forall x_j \in D_j, x_i \in D_i, i \neq j, i, j = \overline{1, 7}, \quad (1)$$

with the initial

$$T(x_j)|_{t=0} = T_0(x_j), \forall x_j \in D_j, j = \overline{1, 7}, \quad (2)$$

and boundary conditions

$$-\lambda(x_j, T) \frac{\partial T}{\partial x_j} \Big|_{x_j \in \Gamma_j \setminus D_j} = q_{\Gamma_j}, i \neq j, i, j = \overline{1, 7}, \quad (3)$$

$$-\lambda(x_j, T) \frac{\partial T}{\partial x_j} \Big|_{x_j \in \Gamma_j \cap (D_i \setminus \Gamma_i)} = \left(\frac{\partial}{\partial x_i} (\lambda(x_i, T) \frac{\partial T}{\partial x_i}) + q_v(x_i, T) \delta(x_i) \right) \Big|_{x_i \in D_i \setminus \Gamma_i \cap \Gamma_j},$$

$$T(x_j)|_{x_j \in \Gamma_j} = T(x_i)|_{x_i \in D_i}, \forall x_j, x_i \in \Gamma_j \cap D_i \setminus \Gamma_i, i \neq j, i, j = \overline{1, 7}, \quad (4)$$

$$\lambda(x_j, T) \frac{\partial T}{\partial x_j} \Big|_{x_j \in \Gamma_j \cap \Gamma_i} = \lambda_i^*(x_i, T) \frac{\partial T}{\partial x_i} \Big|_{x_i \in \Gamma_i \cap \Gamma_j}, i \neq j, i, j = \overline{1, 7}, \quad (5)$$

$$\sum_{j=1,5,6} \left(-\lambda(x_j, T) \frac{\partial T}{\partial x_j} \right) \Big|_{x_j \in \Gamma_1 \cap \Gamma_i \cap \Gamma_6} = 0, \quad (6)$$

$$T(x_i)|_{x_i \in \Gamma_i} = T(x_j)|_{x_j \in \Gamma_j}, \forall x_i, x_j \in \Gamma_i \cap \Gamma_j, i \neq j, i, j = \overline{1, 7}. \quad (7)$$

Each of Eqs. (1) can separately be solved numerically for known initial and boundary conditions by using summary approximation schemes. However, the presence of the component

$\frac{1}{\delta(x_j)} \lambda(x_i, T) \frac{\partial T}{\partial x_i} \Big|_{x_i \in \Gamma_i \cap D_j}$ in (1), which models the thermal influence of the i -th plate on

the state of plates numbered j at the site of their connection, as well as conditions (4)-(7), require the development of a specialized algorithm. Let us note that in the problem formulation under consideration, the boundary conditions are written in the simplest form. The geometric parameters of the connected elements must be taken into account in specific cases.

Underlying the algorithm being developed are principles for constructing locally one-dimensional schemes, as well as methods of solving difference problems by graphs. Let us introduce the coordinate system (Fig. 1b) for each plate in the system, and also the matched spatial network ω_h in the domain $D = \cup D_i, i = \overline{1, 7}$:

$$\omega_h = \{x_i^{(p)} = ih_j^{(p)}, h_j^{(p)} N_j^{(p)} = l_j^{(p)}, p = 1, 2, i = \overline{0, N_j^{(p)}}, j = \overline{1, 7}\}. \quad (8)$$

The algorithm consists of the step-by-step solution of problem (1)-(7). The quantity of steps and their sequence are determined by the structural diagram. The temperature distribution in the plates is investigated in the direction $x_j^{(1)}, j = \overline{1, 7}$ in the first step. As in the case of traditional locally one-dimensional schemes, the following system of inter-related heat-conduction equations given for each layer $x_i^{(2)}, i = \overline{0, N_j^{(2)}}, j = \overline{1, 7}$, in the appropriate graphs (Fig. 2a) is considered:

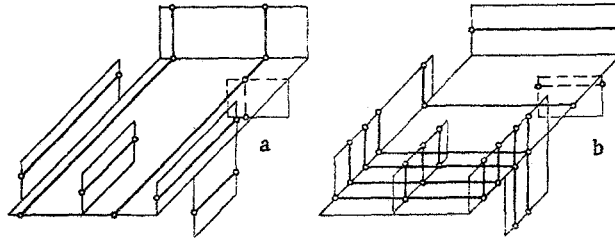


Fig. 2. Graphs used to solve the initial problem in the step (a) in the direction $x_j^{(1)}$, and (b) in the direction $x_j^{(2)}$, $j = \overline{1, 7}$

$$\rho(x_j^{(1)}, T) C_p(x_j^{(1)}, T) \frac{\partial T}{\partial t} = L_1 T(x_j^{(1)}) + q_v(x_j^{(1)}, T),$$

$$\forall x_j \in D_j, j = \overline{1, 7}. \quad (9)$$

$$T(x_j^{(1)})|_h = T(x_j^{(1)})|_{h-1}, T(x_j^{(1)})|_{h=0} = T_0(x_j^{(1)}), j = \overline{1, 7}, \quad (10)$$

$$-\lambda(x_j^{(1)}, T) \frac{\partial T}{\partial x_i^{(1)}} \Big|_{x_j^{(1)} \in \Gamma_j \setminus D_i} = q_{\Gamma_j}, i \neq j, i, j = \overline{1, 7}, \quad (11)$$

$$\lambda(x_1^{(1)}, T) \frac{\partial T}{\partial x_1^{(1)}} \Big|_{x_1^{(1)} \in \Gamma_1 \cap \Gamma_3} = \lambda(x_3^{(1)}, T) \frac{\partial T}{\partial x_3^{(1)}} \Big|_{x_3^{(1)} \in \Gamma_3 \cap \Gamma_1},$$

$$-\lambda(x_4^{(1)}, T) \frac{\partial T}{\partial x_4^{(1)}} \Big|_{x_4^{(1)} \in \Gamma_4 \cap (D_i \setminus \Gamma_i)} = \left(\frac{\partial}{\partial x_1^{(1)}} \left(\lambda(x_1^{(1)}, T) \frac{\partial T}{\partial x_1^{(1)}} \right) + \right.$$

$$\left. + q_v(x_1^{(1)}, T) \right) \delta(x_1^{(1)}) \Big|_{x_1^{(1)} \in (D_i \setminus \Gamma_i) \cap \Gamma_4}, \quad (12)$$

$$T(x_4^{(1)})|_{x_4^{(1)} \in \Gamma_4} = T(x_1^{(1)})|_{x_1^{(1)} \in D_i}, \forall x_4^{(1)}, x_1^{(1)} \in \Gamma_4 \cap (D_i \setminus \Gamma_4), \quad (13)$$

$$T(x_i^{(1)})|_{x_i^{(1)} \in \Gamma_i} = T(x_j^{(1)})|_{x_j^{(1)} \in \Gamma_j}, \forall x_i, x_j \in \Gamma_i \cap \Gamma_j, i = j, i, j = \overline{1, 7}, \quad (14)$$

Let us note that the factorization method can be used to solve this problem in case it is given by the simplest graph, and by the method of solving difference problems by graphs [4] if it is given by a complex graph (the number of edges is ≥ 2).

Furthermore, the temperature distribution in the directions $x_j^{(2)}$, $j = \overline{1, 7}$, is investigated in the second step. In this case a system of interrelated one-dimensional heat-conduction equations determined on one of the graphs displayed in Fig. 2b is solved for each layer $x_j^{(1)}$, $i = \overline{0, N_j^{(1)}}$, $j = \overline{1, 7}$:

$$\rho(x_j^{(2)}, T) C_p(x_j^{(2)}, T) \frac{\partial T}{\partial t} = L_2 T(x_j^{(2)}) + q_v(x_j^{(2)}, T),$$

$$\forall x_j^{(2)} \in D_j, j = \overline{1, 7}, \quad (15)$$

$$T(x_j^{(2)})|_h = T(x_j^{(2)})|_{h-1}, T(x_j^{(2)})|_{h=0} = T_0(x_j^{(2)}), j = \overline{1, 7}, \quad (16)$$

$$-\lambda(x_j^{(2)}, T) \frac{\partial T}{\partial x_i^{(2)}} \Big|_{x_j^{(2)} \in \Gamma_j \setminus D_i} = q_{\Gamma_j}, i \neq j, i, j = \overline{1, 7}, \quad (17)$$

$$\lambda(x_j^{(2)}, T) \frac{\partial T}{\partial x_j^{(2)}} \Big|_{x_j^{(2)} \in (\Gamma_j \cap \Gamma_i) \setminus \Gamma_k} = \lambda(x_i^{(2)}, T) \frac{\partial T}{\partial x_i^{(2)}} \Big|_{x_i^{(2)} \in (\Gamma_i \cap \Gamma_j) \setminus \Gamma_k},$$

$$i \neq j \neq k, i, j, k = 1, 2, 5, 6, \quad (18)$$

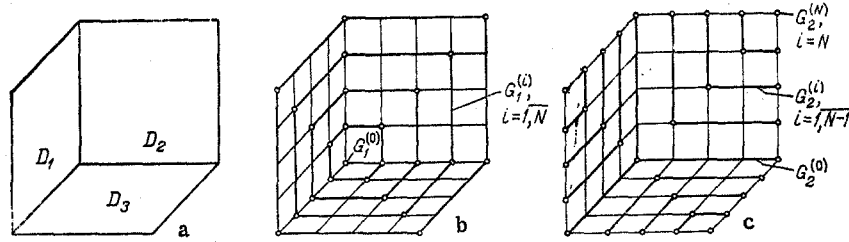


Fig. 3. Orthogonal connection of the plates: a) connection diagram; b) construction of the graph $G_1^{(i)}$, $i = \overline{1, N}$; c) construction of the graphs $G_2^{(0)}$; $G_2^{(i)}$, $i = \overline{1, N}$, $j = 1, 2, 3$.

$$-\lambda(x_7^{(2)}, T) \frac{\partial T}{\partial x_7^{(2)}} \Big|_{x_7^{(2)} \in \Gamma_7 \cap D_1 \setminus \Gamma_1} = \frac{\partial}{\partial x_1^{(2)}} (\lambda(x_1^{(2)}, T) \frac{\partial T}{\partial x_1^{(2)}} + q_v(x_1^{(2)})) \Big|_{x_1^{(2)} \in (D_1 \setminus \Gamma_1) \cap \Gamma_7},$$

$$T(x_7^{(2)}) \Big|_{x_7^{(2)} \in \Gamma_7} = T(x_1^{(2)}) \Big|_{x_1^{(2)} \in D_1}, \quad \forall x_7^{(2)}, x_1^{(2)} \in \Gamma_7 \cap (D_1 \setminus \Gamma_1), \quad (19)$$

$$\sum_{j=1,5,6} \left(-\lambda(x_j^{(2)}, T) \frac{\partial T}{\partial x_j^{(2)}} \Big|_{x_j^{(2)} \in \Gamma_1 \cap \Gamma_i \cap \Gamma_j} \right) = 0, \quad (20)$$

$$T(x_i^{(2)}) \Big|_{x_i^{(2)} \in \Gamma_i} = T(x_j^{(2)}) \Big|_{x_j^{(2)} \in \Gamma_j}, \quad \forall x_i, x_j \in \Gamma_i \cap \Gamma_j, \quad i \neq j, \quad i, j = \overline{1, 7}. \quad (21)$$

The temperature fields in the structure under consideration are determined after completion of the second stage as a result of the numerical solution of problems (9)-(14) and (15)-(21).

This algorithm permits solution of the initial problem (1)-(7) considerably more simply than in the case of applying the "skeleton" structure method. This is because there are no iterations to determine the temperature fields in the "skeleton" structure which would be used to connect the temperature fields of the plates.

Let us examine still another specific case of the orthogonal connection of plates which is widely used in different structures (Fig. 3a). The thermal state of these plates can be determined by using the method of elementary heat balances or the "skeleton" structure method. From the viewpoint of universalization of the thermal computation, we consider the specialized algorithm that realizes the solution of the following boundary-value problem

$$\rho(x_j, T) C_p(x_j, T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_j} \left(\lambda(x_j, T) \frac{\partial T}{\partial x_j} \right) + q_v(x_j, T, t), \quad (22)$$

$$T(x_j) \Big|_{t=0} = T_0(x_j), \quad \forall x_j \in D_j, \quad j = 1, 2, 3, \quad (23)$$

$$-\lambda(x_j, T) \frac{\partial T}{\partial x_j} \Big|_{\Gamma_j} = q_{\Gamma_j}, \quad \forall x_j \in \Gamma_j \setminus D_i, \quad i \neq j, \quad i, j = 1, 2, 3, \quad (24)$$

$$\lambda(x_j, T) \frac{\partial T}{\partial x_j} \Big|_{x_j \in \Gamma_j \cap \Gamma_i} = \lambda(x_i, T) \frac{\partial T}{\partial x_i} \Big|_{x_i \in \Gamma_i \cap \Gamma_j},$$

$$i \neq j, \quad i, j = 1, 2, 3, \quad (25)$$

$$T(x_j) \Big|_{x_j \in \Gamma_j} = T(x_i) \Big|_{x_i \in \Gamma_i}, \quad \forall x_i, x_j \in \Gamma_i \cap \Gamma_j, \quad i \neq j, \quad i, j = 1, 2, 3. \quad (26)$$

Let us introduce a matched space-time network ω_{ht} analogous to (8):

$$\omega_{ht} = \{x_j^{(p)}, i = ih_j^{(p)}; h_j^{(p)} N = l_j^{(p)}, \quad p = 1, 2, \quad i = \overline{0, N},$$

$$j = 1, 2, 3, \quad t = kh_t, \quad h_t N_t = t_{fin}\}. \quad (27)$$

Using the idea of splitting the two-dimensional differential operators in (22), we solve problem (22)-(26) in two steps. In contrast to the algorithms examined earlier, direct application of the method of locally one-dimensional directions (or the method of vari-

able directions in a particular case) is difficult because the sequential examination of temperature fields in plates in appropriate directions results in the total impossibility of matching the time intervals of the solutions of all problems with the time intervals of the temperature field computation in each of the plates.

In the traditional method of locally one-dimensional directions the operator L_2T in (22) is represented in the form of the sum of two operators $L_2T = L_2^{(1)}T + L_2^{(2)}T$ that characterize the heat distribution over the corresponding directions of the plates under consideration. In this case the temperature fields are computed by successive solution of the corresponding one-dimensional equations in the directions $x_j^{(1)}$ and $x_j^{(2)}$.

This sequence is modified somewhat in the method proposed. In the first step the process of heat propagation is considered in the neighborhood of the vertex of the connection as is shown in Fig. 3b. Here the following system of equations is solved, which is given in each graph $G_1^{(i)}$, $i = \overline{1, N}$, encircling the vertex:

$$\rho(x_j^{(p)}, T) C_p(x_j^{(p)}, T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_j^{(p)}} (\lambda(x_j^{(p)}, T) \frac{\partial T}{\partial x_j^{(p)}}) + q_v(x_j^{(p)}, T, t), \quad (28)$$

$$T(x_j^{(p)})|_h = T(x_j^{(p)})|_{h-1}, \quad T(x_j^{(p)})|_{t=0} = T_0(x_j^{(p)}),$$

$$\forall x_j^{(p)} \in D_j, \quad p = 1, 2, \quad j = 1, 2, 3, \quad (29)$$

$$\lambda(x_j^{(1)}, T) \frac{\partial T}{\partial x_j^{(1)}} = \lambda(x_j^{(2)}, T) \frac{\partial T}{\partial x_j^{(2)}},$$

$$T(x_j^{(1)}) = T(x_j^{(2)}), \quad \forall x_j^{(p)} \in D_j \cap G_1^{(i)}, \quad p = 1, 2, \quad i = \overline{1, N}, \quad j = 1, 2, 3, \quad (30)$$

$$\lambda(x_j, T) \frac{\partial T}{\partial x_j} \Big|_{x_j \in \Gamma_j} = \lambda(x_i, T) \frac{\partial T}{\partial x_i} \Big|_{x_i \in \Gamma_i},$$

$$T(x_j)|_{x_j \in \Gamma_j} = T(x_i)|_{x_i \in \Gamma_i}, \quad \forall x_i, x_j \in \Gamma_i \cap \Gamma_j, \quad i \neq j, \quad i, j = 1, 2, 3. \quad (31)$$

A particular case of the graph $G_1^{(i)}$ is the degenerate graph $G_1^{(0)}$ that consists of one vertex, the vertex of the plate connections. The following heat-balance equation is given for this graph

$$C_m^{(0)} \frac{dT}{dt} = q_v^{(0)}, \quad (32)$$

where $C_m^{(0)}$ is the mass specific heat of the vertex, and $q_v^{(0)}$ is the sum of the heat sources, both the external and the internal acting at the vertex.

For $i = 1$ the system of partial differential equations (28)-(31) degenerates into a system of ordinary differential equations describing the temperature change at the vertices of the graph $G_1^{(1)}$.

After having solved the problem (28)-(31), (32), the following system of equations, defined on each graph $G_2^{(0)}, G_2^{(i)}, i = \overline{1, N}, j = 1, 2, 3$ (Fig. 3c) is examined in the second stage:

$$\rho(x_j^{(p)}, T) C_p(x_j^{(p)}, T) \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_j^{(p)}} \left(\lambda(x_j^{(p)}, T) \frac{\partial T}{\partial x_j^{(p)}} \right) + \quad (33)$$

$$+ q_v(x_j^{(p)}, T, t), \quad \forall x_j^{(p)} \in D_j \cap (G_2^{(i)} \cup G_2^{(0)}),$$

$$T(x_j^{(p)})|_{t=0} = T_0(x_j^{(p)}), \quad T(x_j^{(p)})|_h = T(x_j^{(p)})|_{h-1}, \quad (34)$$

$$-\lambda(x_j^{(p)}, T) \frac{\partial T}{\partial x_j^{(p)}} \Big|_{x_j^{(p)} \in \Gamma_j} = q_{\Gamma_j}, \quad (35)$$

$$\lambda(x_j^{(1)}, T) \frac{\partial T}{\partial x_j^{(1)}} = \lambda(x_j^{(2)}, T) \frac{\partial T}{\partial x_j^{(2)}},$$

$$T(x_j^{(1)}) = T(x_j^{(2)}), \quad \forall x_j^{(p)} \in D_j \cap D_{2,j}, \quad i = 1, N, \quad p = 1, 2, \quad (36)$$

$$G_m^{(0)} \frac{dT}{dt} = \sum_{j=1}^3 \lambda(x_j^{(p)}, T) \frac{\partial T}{\partial x_j^{(p)}} + q_v^{(0)}, \quad (37)$$

$$i = 1, N, p = 1, 2, j = 1, 2, 3. \quad (38)$$

Let us note that the system of equations (33) for the graph $G_{(2)}^0$, one of whose vertices coincides with the vertex of the plate connections, is compiled exactly the same as in the method of "skeleton" structures, directly for the plate connection.

To complete the description of the heat propagation in the second stage, the following heat-balance equation, analogous to (32), must be given at the tip points of the plate in the second stage, namely, on the degenerate graphs $G_{2,j}^{(N)}$, $j=1, 2, 3$:

$$C_{mj}^{(N)} \frac{dT}{dt} = q_{vj}^{(N)}, j = 1, 2, 3. \quad (39)$$

Temperature fields in orthogonally connected plates are determined as a result of successive solution of systems (28)-(31), (32) and (33)-(38), (39).

In conclusion, we note that the algorithms examined in this paper as well as in [3] permit substantial simplification of application of the method of "skeleton" structures for the investigation of the thermal regime of a number of structures because of enlargement of the individual modules of the mathematical thermal model, and therefore, enlargement of the model as a whole.

NOTATION

T , temperature; t , time; x, z , spatial coordinates; LT , a differential operator of parabolic type; q_v , a source function; D , domain in which the solution is sought for the appropriate heat-conduction equation; Γ , boundary of the domain D , the subscript of the boundary of the appropriate domain; λ , heat-conduction coefficient; ρ , density; C_p , specific heat; C_m , mass specific heat; l , plate linear dimension; δ , plate thickness; ω_h , spatial network; ω_{ht} , space-time network, h , network spacing; G , a graph; N , number of mesh nodes. Subscripts: i, j, k , coordinates of points or domains; p , direction; k , time; fin , finite time.

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